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The Structure of the Algebra of Observables in the Intermediate Situation of the ϵ-Model

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We look at the action of the spin-1/2 operators of quantum mechanics on the state of an entity in a physical way, and use this as a guideline to define the operators of the intermediate situations of a general spin-1/2 measurement model called the ε-model. Then we test the possible linearity of the operators so constructed.

1. INTRODUCTION

In the hidden measurement approach (Aerts, 1983, 1986, 1987) the probabilities of quantum mechanics are explained as due to the presence of fluctuations on the interaction between the measurement apparatus and the entity under study. The approach was given this name because it considered an experiment as a class of subexperiments (the hidden measurements), parametrized by a real parameter and indistinguishable to the macroscopic observer. The resulting probabilities through the averaging process over the whole class of hidden measurements was shown to coincide with the quantum probabilities. As an example, a model for the spin–1/2 experiments was introduced, which was later generalized to include cases of arbitrary fluctuations, going from maximal fluctuations to zero fluctuations, coinciding respectively with quantum and classical mechanics. The amount of fluctuation was parametrized by a real parameter $\varepsilon \in [0,1]$, hence the name ε-model (Aerts *et al.*, 1993a, b). By varying ε over [0,1] intermediate cases were found, which were neither quantum nor classical. To study these intermediate cases a theory much more general than quantum mechanics was presented (Aerts, 1994; Aerts and Durt 1994a, b). The study was done in several different

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mathematical categories. The structure of the Piron lattice of properties (Piron, 1976) of the entity was found to be Boolean for the case with zero fluctuations and pure quantum for the case of maximal fluctuations. For the intermediate cases the lattice was neither Boolean nor quantum (Aerts and Durt, 1994a, b). For the closures, a disappearing of the superposition principle during the transition from quantum to classical was observed (Aerts and Durt, 1994a, b), and for the intermediate cases it was proven that the Piron axioms to find a representation of the entity in a general Hilbert space are violated. In the category of the probability structures, a transition from Kolmogorovian to non-Kolmogorovian was demonstrated and the intermediate cases were neither Kolmogorovian nor quantum (D. Aerts, 1995; S. Aerts, 1996).

Here we will present the study of the intermediate situations of the ε-model in the category of the observables. The linear operators which are used in quantum mechanics to describe the observables will be investigated in a physical way to get a definition for the operators of the intermediate situations.

2. THE ε-MODEL

As the ε-model has been introduced already in great detail (Aerts *et al.*, 1993a, b), we will only repeat the main characteristics of the model to make this article self-contained. The physical entity S that we consider is a point particle P on the surface *sur f* of a sphere with center O and radius 1, which we will call the Poincaré sphere. The unit vector u where the particle is located on *surf* represents the state p_v of the particle. With every unit vector u an experiment e_u^ε is associated with a set of outcomes $O_{e_u^\varepsilon} = \{+1, -1\}$ such that the entity in case of an outcome $+1$ ends up at u after the measurement and at $-u$ in case of an outcome -1. The probabilities with which these outcomes occur for the entity in a state p_v are denoted by $P^\varepsilon(p_u|p_v)$ for the outcome $+1$ and by $p^\varepsilon(p_{-u}|p_v)$ for the outcome -1. The parameter $\varepsilon \in [0, 1]$. The probabilities are given by:

1. $\varepsilon \leq v \cdot u$: $P^\varepsilon(p_u|p_v) = 1$.
2. $-\varepsilon \leq v \cdot u \leq +\varepsilon$: $P^\varepsilon(p_u|p_v) = (1/2\varepsilon)(v \cdot u + \varepsilon)$.
3. $v \cdot u \leq -\varepsilon$: $P^\varepsilon(p_u|p_v) = 0$.

$P^\varepsilon(p_{-u}|p_v) = 1 - P^\varepsilon(p_u|p_v)$ for all three cases. For $\varepsilon = 0$, the experiment e_u^ε gives for the entity on the equator an outcome $+1$ with probability 1/2 and an outcome -1 with probability 1/2. In the case $\varepsilon = 1$, the probabilities coincide with the probabilities of a spin-1/2 entity in quantum mechanics, such that the entity S can be described in a Hilbert space and the experiments e_u^ε by the linear self-adjoint operators of that Hilbert space. When we vary ε over [0, 1], we get intermediate cases going from quantum ($\varepsilon = 1$) to classical ($\varepsilon = 0$). It has been proven that only for $\varepsilon = 1$ are the axioms to derive a Hilbert space structure from the lattice of properties satisfied. Because no Hilbert space structure is available for the intermediate cases, we need other guidelines to study the ε-model in the category of the operators, namely averages of physical observables.

3. THE REPRESENTATION OF SPIN-1/2 STATES ON THE POINCARÉ SPHERE

First we establish a correspondence between the points on the Poincaré sphere *surf* and the eigenvectors of the spin-1/2 operators in the complex space C^2, in the following way. The mapping

$$S_U: \quad u = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \to S_u = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix}$$

maps a unit vector u on the spin-1/2 operator S_u which has two orthogonal eigenvectors that form a basis for the complex space C^2, namely

$$s_{u+} = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\varphi/2} \\ \sin\frac{\theta}{2}\, e^{i\varphi/2} \end{pmatrix}, \qquad s_{u-} = \begin{pmatrix} -\sin\frac{\theta}{2}\, e^{-i\varphi/2} \\ \cos\frac{\theta}{2}\, e^{i\varphi/2} \end{pmatrix}$$

with eigenvalue $+1$ and -1, respectively. We can attribute the following meaning to these eigenvectors: if the entity is in a state s_{u+}, we will find the value $+1$ with certainty. The interpretation is then that on the Poincaré sphere the entity is in the state p_u given by the point u. In short, we make a unit vector u of *surf* correspond to an eigenvector s_{u+} in C^2.

This correspondence is one-to-one. On the Poincaré sphere we have two degrees of freedom: θ and φ. In the complex space C^2 there are four: each complex number can be written as the sum of its real part and its imaginary part; but by demanding that the norm of the eigenvector is 1, and because an eigenvector is defined up to an arbitrary constant (by the first requirement of modulus 1) we have indeed a one-to-one correspondence between the vector of unit length u and the set of eigenvectors of S_u with eigenvalue $+1$ and norm 1.

4. THE SPIN-1/2 OPERATORS FOR THE INTERMEDIATE SITUATIONS

4.1. How to Generalize Quantum Operators

Insofar as there is a connection between an element of the complex space C^2 and an element of the Poincaré sphere, there exists a connection between the operators in the complex space C^2 and the operators on the Poincaré sphere. Our guidelines will be: (1) that the averages of a physical observable are independent of our description (thus independent of the fact that we are describing it in the complex space C^2 or on the Poincaré sphere) and (2) the connection that exists between the product in C^2 and the scalar product of vectors on the Poincaré sphere. More precisely, the average $S_1(\psi_w)$ of an observable S_1 when the entity is in a state $\psi_w \in C^2$ is given by

$$\overline{S_1(\psi_w)} = \langle S_1(\psi_w)|\psi_w\rangle$$

and by denoting the action of the spin operator on the Poincaré sphere by T_1 [thus with $S_1(\psi_w) \in C^2$ is associated the vector $T_1(w)$ on the Poincaré sphere] we can write the following well-known connection between the products:

$$|\langle S_1(\psi_w)|\psi_w\rangle|^2 = \frac{1 + T_1(w) \cdot w}{2}$$

These two formulas are sufficient to define and study the action on a state w by the operator T_ε associated with a general spin-1/2 measurement e_u^ε on the Poincaré sphere. By the word "action" we mean that the square of the average $T_\varepsilon(w)$ of the operator T_ε on the Poincaré sphere when the entity is in a state p_w is given per definition and in analogy with the quantum case by

$$\overline{T_\varepsilon(w)}^2 = \frac{1 + T_\varepsilon(w) \cdot w}{2}$$

The operator T_1 corresponds to a rotation over π. This is so because the formula shows us that the angle between $T_1(w)$ and w is twice the angle θ between w and u:

$$\frac{1 + T_1(w) \cdot w}{2} = \overline{T_1(w)}^2 = \cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

because the average $\overline{T_1(w)}$ is $\cos\theta$ in the quantum case. Moreover, also the angle between $T_1(w)$ and u is θ:

$$\langle S_1(\psi_w)|\psi_u\rangle = \langle\psi_w|S_1^\top(\psi_u)\rangle = \langle\psi_w|S_1(\psi_u)\rangle = \langle\psi_w|\psi_u\rangle$$

because S_1 is self-adjoint and ψ_u is an eigenvector of S_1, with eigenvalue $+1$. Using elementary triangle inequalities on the sphere, we then see that S_1 is indeed a rotation over π.

Clearly these physical assumptions only make it possible to define the angle γ_w between a vector w and its image $T_\varepsilon(w)$. In general this means that we only can say that $T_\varepsilon(w)$ lies on a small circle on the Poincaré sphere centered around the axis $[-w, w]$ and making an angle γ_w with w. To define our $T_\varepsilon(w)$ unambiguously we will make the mathematical assumption that

$T_\varepsilon(w)$ makes the same angle θ with u as w does. This is inspired by the fact that in the quantum case we have that $(T_1)^2 = id_{surf}$. If we need the possibility to get back from $T_\varepsilon(w)$ to w by applying the operator a second time we have to be aware of the fact that the action of T_ε on $T_\varepsilon(w)$ depends on the angle $\theta_{T_\varepsilon(w)}$ between $T_\varepsilon(w)$ and u, and it is necessary that this angle is equal to the angle θ_w between u and w, if we want that a same but opposite action is possible.

4.2. Construction of the Operators for the Intermediate Situations

We will study the spin operator for the various values of ε. For $\varepsilon = 1$ we have found that the action of the spin operator is that of a rotation of π. Because, as mentioned earlier, the state space is in general not a Hilbert space, we use the following correspondence:

$$\frac{1 + T_\varepsilon(w) \cdot w}{2} = \overline{T_\varepsilon(w)}^2$$

The average $\overline{T_\varepsilon(w)}$ can be calculated by means of the given probabilities:

$$\overline{T_\varepsilon(w)} = (+1)P^\varepsilon(p_u|p_w) + (-1)P^\varepsilon(p_{-u}|p_w)$$

So that we find that, if we make the assumption that $T_\varepsilon(w)$ is of unit length, we can calculate the angle γ_w between $T_\varepsilon(w)$ and w:

1. If $\cos\theta \geq \varepsilon$ or $\cos\theta \leq -\varepsilon$, then $\cos\gamma_w = 1$.
2. If $\varepsilon \geq \cos\theta \geq -\varepsilon$, then $\cos\gamma_w = (2\cos^2\theta - \varepsilon^2)/\varepsilon^2$.

So a vector v on the equator of u will always be mapped onto its antipode $-v$, and u will always be mapped by T_ε onto itself. More precisely, if we define the eigensets $\text{eig}_u^\varepsilon(+1)$ and $\text{eig}_u^\varepsilon(-1)$ as the sets $\text{eig}_u^\varepsilon(+1) = \{p_v | \varepsilon \leq v \cdot u\}$ and $\text{eig}_u^\varepsilon(-1) = \{p_v | v \cdot u \leq -\varepsilon\}$, then every vector of an eigenset will be unchanged by the operator T_ε.

It is obvious that this indeed reduces to a rotation over π if $\varepsilon = 1$. This is so because the formula shows us that the angle γ_w between $T_1(w)$ and w is twice the angle θ between w and u. Moreover, according to the mathematical assumption, the angle between $T_1(w)$ and u is also θ.

4.3. The Azimuthal Change

Since we have demanded that the angle between $T_\varepsilon(w)$ and u should be equal to the angle between w and u, we can look at what happens with the other remaining degree of freedom and calculate the "azimuthal angle" $\varphi(\theta, \varepsilon)$ between $T_\varepsilon(w)$ and w, for w not in an aigenset, where it would be unchanged by the operator, as mentioned earlier. By "azimuthal angle"

$\varphi(\theta, \varepsilon)$ we mean the difference between the angles φ_2 and φ_1 if we write $w = (\sin\theta\cos\varphi_1, \sin\theta\sin\varphi_1, \cos\theta)$ and $T_\varepsilon(w) = (\sin\theta\cos\sin\varphi_2, \sin\theta\,\varphi_2, \cos\theta)$. After a small calculation we find that

$$\varphi(\theta, \varepsilon) = 2\arcsin\left[\sqrt{\frac{1}{\sin^2\theta} - \frac{\cot^2\theta}{\varepsilon^2}}\right]$$

And the differentiations of $\varphi(\theta, \varepsilon)$ with respective to θ and ε are, respectively,

$$\frac{d\varphi(\theta, \varepsilon)}{d\theta} = \frac{2|\cos\theta|}{\cos\theta\sqrt{\varepsilon^2 - \cos^2\theta}}\,\frac{\sqrt{1-\varepsilon^2}}{\sin\theta}$$

$$\frac{d\varphi(\theta, \varepsilon)}{d\varepsilon} = \frac{2|\cos\theta|}{\cos\theta\sqrt{\varepsilon^2 - \cos^2\theta}}\,\frac{\cos\theta}{\varepsilon\sqrt{1-\varepsilon^2}}$$

For $\varepsilon = 1$ we see that $d\varphi(\theta, \varepsilon)/d\theta = 0$, $\forall\theta \in [0, \pi]$, so that $\varphi(\theta, 1) = c_1 = \varphi(\pi/2, 1) = \pi$. The cases $\theta = 0$ and $\theta = \pi$ are irrelevant, since for them φ is undefined. For $\theta = \pi/2$ we find that $d\varphi(\theta, \varepsilon)/d\varepsilon = 0$, $\forall\varepsilon \in [0, 1]$, so that $\varphi(\pi/2, \varepsilon) = c_2 = \varphi(\pi/2, 1/2) = \pi$. Moreover, for $\varepsilon = 1$ we also have that $\varphi(\pi/2, 1) = \pi$. Thus $\varphi(\pi/2, \varepsilon) = \pi$, $\forall\varepsilon \in [0, 1]$.

The troublesome case $\varepsilon = 0$ needs some special attention. There $\varphi(\theta, \varepsilon)$ is undefined for $\theta = \pi/2$ [w lies in an eigenset if $\theta_w \neq \pi/2$, so that $T_0(w) = w$] and we will need some special considerations to decide whether we should use the extrapolation of the eigensets (so that $T_0 = id$), or whether we should maintain the rotation over π for the equator as in all other cases of ε. A strong argument for the first choice would be that we then have a linear operator just as in the quantum case; but this "necessity" of linearity of the operator is refuted by an earlier result which states that for the intermediate situations we lose linearity of the operator (Aerts and D'Hooghe, 1996). Because the proof is very short, we will repeat it here briefly.

4.4. Linearity or Nonlinearity

For $\varepsilon = 1$ we see that the operator T_ε is linear:

$$T_1\left[\frac{w + v}{\sqrt{2}}\right] = \frac{T_1(w) + T_1(v)}{\sqrt{2}}$$

This is trivial if we write the action of T_1 in some more geometrical way, making clear its linearity: $T_1(w) = -w + 2(u\cdot w)u$.

For the intermediate cases ($0 \neq \varepsilon \neq 1$) linearity is no longer available.

Theorem. T_ε is a linear operator if and only if $\varepsilon = 1$.

Proof. We have to show that $T_\varepsilon(w + v) = T_\varepsilon(w) + T_\varepsilon(v)$ for every v, w on the sphere. If we suppose that T_ε is a linear operator on the Poincaré sphere, we have that

$$\frac{T_\varepsilon(w) + T_\varepsilon(v)}{\sqrt{2}} = T_\varepsilon\left[\frac{w + v}{\sqrt{2}}\right]$$

Let us take $w = u$ and v an arbitrary vector on the equator of u. Then $T_\varepsilon(w) = w$, and $T_\varepsilon(v) = -v$, and we find that

$$\frac{T_\varepsilon(w) + T_\varepsilon(v)}{\sqrt{2}} = \frac{w - v}{\sqrt{2}}$$

We have now two possibilities for $(w + v)/\sqrt{2}$

1. If $(w + v)/\sqrt{2}$ is in an eigenset,

$$T_\varepsilon\left[\frac{w + v}{\sqrt{2}}\right] = \frac{w + v}{\sqrt{2}}$$

and T_ε is clearly not linear.

2. $(w + v)/\sqrt{2}$ is not in an eigenset. The angle θ between $(w + u)/\sqrt{2}$ and u is $\pi/4$. So we find

$$\cos \gamma_{(w+v)/\sqrt{2}} = \frac{2\cos^2\theta - \varepsilon^2}{\varepsilon^2} = \frac{1 - \varepsilon^2}{\varepsilon^2}$$

On the other hand, we see that the angle $\gamma_{(w+v)/\sqrt{2}}$ between $(w - v)/\sqrt{2}$ and $(w + v)/\sqrt{2}$ is $\pi/2$ such that we also have that $\cos \gamma_{(w+v)/\sqrt{2}} = 0$. This can be the case if and only if $\varepsilon = 1$. Thus only in the quantum case do we find that T_ε is linear.

4.5. Return to the Classical Case

Now we can return to the classical case. The average $\overline{T_0(w)}$ for an arbitrary w on the equator ($w \cdot u = 0$) is given by

$$\overline{T_0(w)} = (+1)P^0(p_u|p_w) + (-1)P^0(p_{-u}|p_w) = (+1)\,\frac{1}{2} + (-1)\,\frac{1}{2} = 0$$

Hence $[1 + T_0(w) \cdot w]/2 = 0$, and the angle γ_w between w and $T_0(w)$ is π.

We have two possibilities:

1. We define T_0 as the identity for $\varepsilon = 0$, making an extrapolation of the eigensets (the upper and lower open half-spheres). Making the eigensets closed by defining the action of T_0 on the equator as the identity, we recover

a linear operator. But then the eigensets get mixed up for the unnecessary property of linearity.

2. We define T_0 as the identity on the whole sphere, except on the equator, where it maps the points onto their antipodes. So there is no linearity in the classical case, too. By making this choice we maintain our foregoing guidelines and respect symmetry. Another reason is that for a state on the equator we have an instability in the classical case: where the states of the upper and the lower open half spheres have a predetermined outcome, only a probability is given on the equator. So we choose for the classical case on the equator the same action for the operator as in the quantum case, namely a rotation over π, because their probabilities are the same.

5. CONCLUSION

We have constructed the set of operators for the intermediate situations of the ε-model and have shown that only for the case with maximal fluctuations of the interaction between entity and measurement apparatus are the operators linear on the Poincaré sphere. This result suggested that we define the action of the operator on the equator in the classical case as a rotation over π.

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